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Theory of Dispersion in Microstrip of Arbitrary Width

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***Abstract*—An analytic theory for the dispersion of the fundamental mode on wide open microstrip is presented. Only a single basis function is needed to accurately represent each of the change and current distributions on the strip, thus allowing more efficient determination of the propagation constant as compared to moment-method solutions requiring a larger number of basis functions. The results obtained blend smoothly into results of high-frequency (Wiener–Hopf) theories, and still retain the appealing physical interpretation in terms of capitance and inductance of the narrow strip theory previously obtained by the authors.**

I. Introduction

In PREVIOUS work, the authors [1] have presented an analytic theory of dispersion for narrow open microstrip (that is, for which the strip is small compared to substrate thickness) in terms of a dispersive series inductance and capitance, generalizing the classical expression for the propagation constant from transmission line theory which involves the static values of these parameters. Because an accurate form for the current and charge distributions (which are the same for this case) was available, it was possible to avoid more cumbersome moment function expansions, and to obtain a relatively simple dispersion relation possessing the clear physical interpretation referred to above. In reviewing numerical results available in the literature for wider microstrip, whose strip width is comparable to substrate thickness, the authors found significant discrepancies between workers who used different methods to attack the problem [2]. The best methods seem to be those which can represent the current and charge distributions (especially the edge singularities) accurately with a minimum number of basis functions.

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The goal of the present study is to formulate an analytic theory of dispersion similar to [1] which will be valid for wider strips, yet still retain both analytical and computational straightforwardness as well as clear physical insight into the problem. Crucial to this is the recognition that the charge and current distributions now differ significantly from those in the narrow-strip limit, and also differ to some extent from each other. Thus an important part of the discussion depends on having accurate and reasonably simple functional descriptions of these distributions. The results will be examined to see what degree the difference of these distributions from the narrow-strip case and from each other affects the accuracy of the computed dispersion curves.

Of published numerical work, references [3]–[5] offer results that we might classify as applying to "wide" microstrip, and these will be used as the basis for comparison. Also, although we shall consider strips wide compared to the substrate, the strips are not allowed to become *electri-*

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cally large ($k_0 l \lesssim 1$), because in this range of parameters, the physical mechanisms are basically different. These are best treated by the methods of [6]–[9] (which use adaptations of the more appropriate Wiener–Hopf technique).

II. Derivation of the Dispersion Relation Using Static Charge and Current Distributions

We shall proceed from the formulation of [1], making some minor changes in notation. For an assumed propagation factor of $\exp(i\omega t - ik_0\alpha x)$, where α is the (yet unknown) propagation constant of the fundamental mode, normalized to the wavenumber k_0 of free space, and x is distance along the length of the strip (see Fig. 1), we have

$$\int_{-l}^{l} G_e(y-y')\rho(y')dy' = \cosh\sqrt{\alpha^2-1}\; k_0 y, \qquad |y| \leqslant l. \tag{1}$$

Here $\rho(y)$ is the charge distribution on the strip ($-l \leqslant y \leqslant l$) which is considered to be vanishingly thin, while

$$G_e(y) = 2\int_0^\infty \frac{(u_n \tanh u_n T)\cos k_0\lambda y}{\epsilon_r u_0 + u_n \tanh u_n T}\,\frac{d\lambda}{u_0}. \tag{2}$$

The quantity $T = k_0 t$ is the substrate thickness normalized to the free space wavenumber, and

$$u_n(\lambda^2+\alpha^2-\mu_r\epsilon_r)^{1/2}, \qquad u_0 = (\lambda^2+\alpha^2-1)^{1/2}, \qquad \mathrm{Re}(u_0) \geqslant 0 \tag{3}$$

where ϵ_r and μ_r are, respectively, the relative permittivity and permeability of the substrate. It might be mentioned that the quantity α^2 is usually denoted by ϵ_{eff} or $\epsilon_{r_{\text{eff}}}$, the effective dielectric constant of this mode.

Once the solution of (1) is known as a function of α, the longitudinal current density $J_x(y)$ is then found from

$$\int_{-l}^{l} G_m(y-y')J_x(y')dy' = \alpha c_0\left[\cosh\sqrt{\alpha^2-1}\; k_0 y + \int_{-l}^{l} M(y-y')\rho(y')dy'\right] \tag{4}$$

where $c_0 = (\mu_0\epsilon_0)^{-1/2}$ is the velocity of light in free space, while

$$G_m(y) = 2\mu_r\int_0^\infty \frac{\cos k_0\lambda y}{\mu_r u_0 + u_n \coth u_n T}\,d\lambda \tag{5}$$

$$M(y) = 2(\mu_r\epsilon_r - 1)\cdot\int_0^\infty \frac{\cos k_0\lambda y\, d\lambda}{(\epsilon_r u_0 + u_n \tanh u_n T)(\mu_r u_0 + u_n \coth u_n T)u_0}. \tag{6}$$

The solutions $\rho(y)$ and $J_x(y)$ thus obtained, both functions of α, are then inserted into

$$\int_{-l}^{l} [\alpha J_x(y) - c_0\rho(y)]\,dy = 0 \tag{7}$$

(which follows from conservation of charge and the requirement that the transverse current density vanish at the edges of the strip) and (7) then becomes a characteristic equation for determining α. So far, these equations are exact.

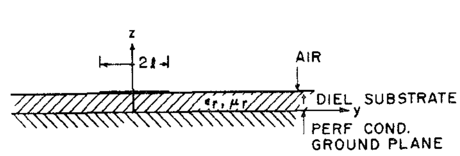

Fig. 1. Open microstrip.

Let us examine the static limit of these equations. We find that $\cosh\sqrt{\alpha^2-1}\; k_0 y \to 1$, and upon changing variables from $k_0\lambda \to \lambda$:

$$G_e \to G_e^{(0)}(y) = 2\int_0^\infty \frac{\tanh \lambda t}{\epsilon_r + \tanh \lambda t}\cos\lambda y\,\frac{d\lambda}{\lambda} \tag{8}$$

$$G_m \to G_m^{(0)}(y) = 2\mu_r\int_0^\infty \frac{\cos\lambda y}{\mu_r + \coth\lambda t}\,\frac{d\lambda}{\lambda} \tag{9}$$

and $M \to 0$ because of $\coth u_n T$ in its denominator, so that (1) and (4) decouple into

$$\int_{-l}^{l} G_e^{(0)}(y-y')\rho(y')dy' = 1, \qquad |y| \leqslant l \tag{10}$$

and

$$\int_{-l}^{l} G_m^{(0)}(y-y')J_x(y')dy' = \alpha c_0, \qquad |y| \leqslant l. \tag{11}$$

Now, the equations which determine the static charge and current distributions on this structure are (see, e.g., [12])

$$\frac{1}{2\pi}\int_{-l}^{l} G_m^{(0)}(y-y')\rho^{(0)}(y')dy' = V \int_{-l}^{l} \rho^{(0)}(y)dy = C_s V \tag{12}$$

$$\frac{1}{2\pi}\int_{-l}^{l} G_m^{(0)}(y-y')J_x^{(0)}(y')dy' = L_s I \qquad \int_{-l}^{l} J_x^{(0)}(y)dy = I \tag{13}$$

where $\rho^{(0)}(y)$ and $J_x^{(0)}(y)$ are the static distributions, V and I are, respectively, the voltage and total current on the strip, and C_s is the distributed capacitance (normalized to ϵ_0) and L_s the distributed inductance (normalized to μ_0) per unit length of the line. From (7) and (10)–(13) we can obtain the well-known expression

$$\alpha_s^2 = \epsilon_{s_{\text{eff}}} = L_s C_s \tag{14}$$

as the static value of ϵ_{eff}. Many computations of L_s and C_s are available in the literature; the authors have found the expressions derived in [10] for the case $\mu_r = 1$ to be particularly convenient. These are accurate to better than 0.75 percent for all values of l/t, and are quoted in Appendix A.

The static distributions $\rho^{(0)}(y)$ and $J_x^{(0)}(y)$ can be expected to be reasonably accurate for nonzero frequencies as well, if neither the strip width nor the substrate thickness becomes electrically large ($k_0 l \lesssim 1$, $k_0 t \lesssim 1$). For definiteness, let us denote by $\rho^{(0)}(y)$ and $J_x^{(0)}(y)$ the solutions of (12) and (13) with $C_s V = 1$ and $I = 1$ from here on. We assume that dispersion can be accounted for by introducing amplitude factors $\rho_0(\alpha)$ and $J_0(\alpha)$:

$$\rho(y) \simeq \rho_0(\alpha)\rho^{(0)}(y) \qquad J_x(y) \simeq J_0(\alpha)J_x^{(0)}(y). \tag{15}$$

The amplitude factors are to be found by substituting (15) into (1) and (4), multiplying (1) by $\rho^{(0)}(y)$ and (4) by $J_x^{(0)}(y)$ and integrating from $-l$ to l, and finally making use of (7). These steps are carried out in Appendix B, and result in

$$\alpha^2 = L(\alpha)C(\alpha) \tag{16}$$

where $L(\alpha)$ and $C(\alpha)$ are a (normalized) dispersive inductance and a (normalized) dispersive capacitance per unit length of the line, respectively. Each consists of a "series" combination of the static part and a dispersive part:

$$L(\alpha) = L_s + L_d(\alpha) \tag{17}$$

$$\frac{1}{C(\alpha)} = \frac{1}{C_s} + \frac{1}{C_d(\alpha)} \tag{18}$$

where $L_d(\alpha)$ and $1/C_d\alpha$ are given by (B.22) and (B.23).

It will be noted that the procedure leading to (16)–(18) is in essence a moment method, where only a single basis function is used to approximate each of the charge and current distributions. The anticipated success of our approach lies in the fact that $\rho^{(0)}(y)$ and $J_x^{(0)}(y)$ provide greater accuracy than any of the single basis functions such as are used in other moment-method approaches (e.g., [4] and [5]). We can therefore compute the dispersion equation more efficiently because fewer basis functions are needed.

We might also have chosen to follow one of the variational approaches available in the literature. Rather simple stationary formulas for L_s and C_s can be found [11]–[14], but for nonzero frequency, the stationary functionals are more complicated, as can be seen in [2], [14]–[16]. Some simplification occurs if transverse currents on the strips are assumed to be negligible. A dispersion relation, also in the form of (16)–(18), can be obtained from the variational expression given in [16], although different expressions for $L_d(\alpha)$ and $1/C_d(\alpha)$ are obtained. However, it can be shown that these two equations differ only in terms which would be absent if transverse currents had been neglected. Numerical results from both methods were compared using $l/t = 2.34$, $k_0 t = 0.107$, and $\epsilon_r = 9.9$ and found to give $\epsilon_{r_{eff}} = \alpha^2$ as 8.339 and 8.337, respectively. This might be expected, since the "moment method" leading to (16)–(18) can also be shown to possess a variational property [18]. Omitting the contribution due to transverse currents (i.e., the difference in functional form between $\rho^{(0)}(y)$ and $J_x^{(0)}(y)$) also yielded 8.337, apparently indicating that this difference has little effect on results. However, as may be concluded from the comparisons in [2], the difference between $\rho^{(0)}(y)$ and $J_x^{(0)}(y)$ and the corresponding expressions for the narrow strip can be quite important, especially for $l/t > 1$.

III. Numerical Results and Discussion

A. Narrow Strips

It can be shown, using the limiting form of the Legendre functions for argument equal to unity [19] and the limiting forms for the elliptic integrals as k_e and $k_m \to 0$, that the dispersion equations derived here pass over into those obtained for narrow strips [1], but the present theory is valid for strips of arbitrary width. Results have been computed for narrow strips and compare

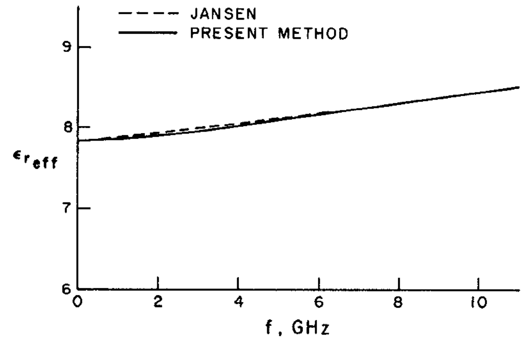

Fig. 2. Effective dielectric constant $\epsilon_{r_{eff}} = \alpha^2$ for open microstrip: $t =$ 0.64 mm, $= 1.5$ mm, $\epsilon_r = 9.9$ as computed by Jansen [5] and by present method.

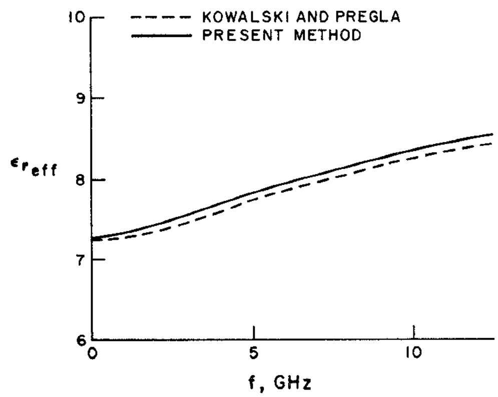

Fig. 3. Effective dielectric constant $\epsilon_{r_{eff}} = \alpha^2$ for open microstrip: $t =$ 1.27 mm, $l = 1.905$ mm, $\epsilon_r = 9.7$ as computed by Kowalski and Pregla [3] and by the present method.

quite well with those of [1], although of course the latter does not require evaluations of Legendre functions and is altogether more appropriate to the task.

B. Wider Strips

Of the results available for wider strips, those of [5] seem to have the greatest likelihood of accuracy. As argued in [2], the moment method used in [5] used a set of basis functions to describe the currents which possess the proper singular behavior at the edges of the strip, and a sufficient number of these is employed to assure an accurate result. Fig. 2 gives a comparison between the results for the widest strip from [5] and from the present method (the various dispersion relations mentioned at the end of the previous section gave indistinguishable results when displayed graphically—this was true for all results presented here). The agreement is nearly exact: the discrepancy is at least as much as the error involved in reading data from the graph in [5]. Kowalski and Pregla [3] have used a variational approach, but use only the current distribution appropriate to a narrow strip as a trial function. While, as seen in [2], this gives good results even for strips as wide as the substrate thickness, a comparison of their results for a wider strip with those of our method (Fig. 3) shows that the narrow strip current distribution is

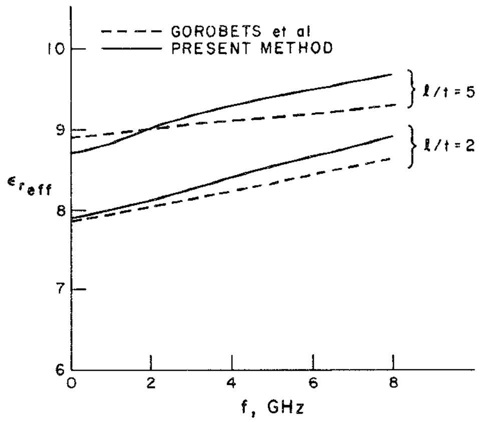

Fig. 4. Effective dielectric constant $\epsilon_{r_{eff}}=\alpha^2$ for open microstrip: $t=1.27$ mm, $\epsilon_r=10.2$; as computed by Gorobets *et al.* [4] and the present method.

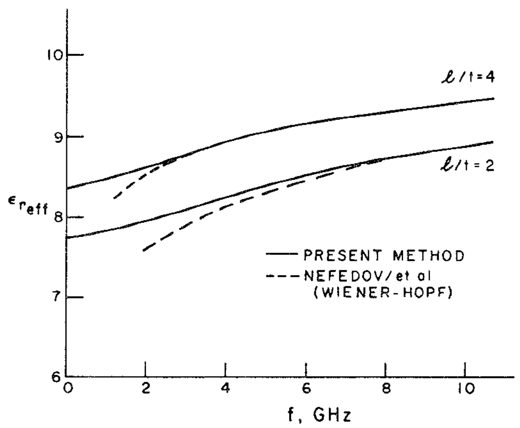

Fig. 5. Effective dielectric constant $\epsilon_{r_{eff}}=\alpha^2$ for open microstrip: $t=1.27$ mm, $\epsilon_r=10$; as computed by Nefedov *et al.* [9] and the present method.

no longer adequate, although the same general trend for the effective dielectric constant is predicted.

In [4], results for very wide strips are computed by what is also (in essence) a moment-method technique, but using a constant distribution of the current on the strip. Comparing results with those of our method in Fig. 4, we see that for $l/t=2$, their method seems to predict a reasonable value for α^2 in the static limit, but dispersion effects are considerably underestimated. For a very wide strip with $l/t=5$, no consistent pattern of error seems to be present. A possible explanation of this is that in both methods, Sommerfeld integrals like (B.24)–(B.26) must be evaluated with rapidly oscillating integrands (the conical functions oscillate more rapidly with τ as the argument is increased); a similar rapid oscillation occurs in [4] due to trigonometric functions. In support of our result for the static limit, we can offer agreement with the graphically displayed results of Wheeler [20] and many others who have studied this case, at least to within the readability of the graphs used for comparison.

Displayed in Fig. 5 are the results of Nefedov *et al.* [9], who apply a Wiener–Hopf technique appropriate to very wide strips and rather high frequencies. It can be seen that in both instances good agreement with our result is obtained at frequencies for which $\sqrt{\epsilon_r}\, k_0 l \gtrsim 0.5$. The agreement with [9] is particularly gratifying since it indicates that the entire range of frequencies can be covered with the present method (for lower frequency) and the Wiener–Hopf approach (for higher frequency), with a considerable region of overlap where both are accurate.

We should note that the current and charge distributions *themselves*, not just the differences between them, influence the character of the dispersion curves. As seen in [1], for example, the dispersion of a narrow strip (for which these distributions have the same form) is actually more pronounced, in general than that of the wide strips presented here.

IV. Conclusion

It has been found that accurate results for the dispersion of open microstrip of arbitrary width can be obtained using only a single basis function each for the charge and current distributions on the strip. Computing times can be considerably shortened compared to moment-method approaches requiring larger numbers of basis functions to represent these quantities. A smooth transition has been observed between this, low-frequency theory, and the higher frequency (Wiener–Hopf) approaches existing in the literature.

Appendix A

In this Appendix, we quote without derivation the closed-form expressions for L_s and C_s obtained in [10]:

$$\frac{1}{C_s}=\frac{1}{2(\epsilon_r+1)}\frac{K(k_e')}{K(k_e)}-\frac{t^2}{4\pi\epsilon_r^2 l^2}\ln\left[1+\frac{4l^2}{a_e t^2}\right] \tag{A.1}$$

$$L_s=\frac{K(k_m')}{4K(k_m)}-\frac{t^2}{4\pi l^2}\ln\left[1+\frac{4l^2}{a_m t^2}\right]. \tag{A.2}$$

Here $K(k)$ is the complete elliptic integral of the first kind, the moduli k_e and k_m are defined in (B.21), while the constants a_e and a_m are given by

$$a_e=-\frac{(\epsilon_r+1)/\epsilon_r^2}{Q(-\delta_\epsilon)+\ln[\pi\epsilon_r/2(\epsilon_r+1)]} \tag{A.3}$$

$$a_m=\frac{2}{\ln(4/\pi)}\simeq 8.2794 \tag{A.4}$$

where

$$\delta_\epsilon=(\epsilon_r-1)/(\epsilon_r+1)$$

and

$$Q(x)=\sum_{m=1}^{\infty}x^m\ln\left(\frac{m+1}{m}\right). \tag{A.5}$$

Appendix B

In this Appendix, we carry out the steps leading to the dispersion relation (16). Substituting (15) into (1) and (4), we have

$$\rho_0(\alpha)\int_{-l}^{l} G_e(y-y')\rho^{(0)}(y')dy' = \cosh\sqrt{\alpha^2-1}\; k_0 y \tag{B.1}$$

$$J_0(\alpha)\int_{-l}^{l} G_m(y-y')J_x^{(0)}(y')dy' = \alpha c_0\left[\cosh\sqrt{\alpha^2-1}\; k_0 y + \rho_0(\alpha)\cdot\int_{-l}^{l} M(y-y')\rho^{(0)}(y')dy'\right]. \tag{B.2}$$

Multiplying (B.1) by $\rho^{(0)}(y)$, and (B.2) by $J_x^{(0)}(y)$, and integrating, we obtain

$$\rho_0(\alpha)\int_{-l}^{l}\int_{-l}^{l} G_e(y-y')\rho^{(0)}(y')\rho^{(0)}(y)dy'dy = \int_{-l}^{l}\rho^{(0)}(y)\cosh\sqrt{\alpha^2-1}\; k_0 y dy \tag{B.3}$$

$$J_0(\alpha)\int_{-l}^{l}\int_{-l}^{l} G_m(y-y')J_x^{(0)}(y')J_x^{(0)}(y)dy'dy = \alpha c_0\left[\int_{-l}^{l} J_x^{(0)}(y)\cosh\sqrt{\alpha^2-1}\; k_0 y dy + \rho_0(\alpha)\cdot\int_{-l}^{l}\int_{-l}^{l} M(y-y')\rho^{(0)}(y')J_x^{(0)}(y)dy'dy\right]. \tag{B.4}$$

Now, by writing $G_e = G_e^{(0)} + \Delta G_e$ and $G_m = G_m^{(0)} + \Delta G_m$, where the static kernels $G_e^{(0)}$ and $G_m^{(0)}$ are given in (8) and (9), we can make use of (12), (13) to simplify part of the left sides of (B.3) and (B.4), recalling that we have set $C_s V$ and I equal to unity in those equations:

$$\rho_0(\alpha)\left\{\frac{2\pi}{C_s} + \int_{-l}^{l}\int_{-l}^{l}\Delta G_e(y-y')\rho^{(0)}(y')\rho^{(0)}(y)dy'dy\right\} = \int_{-l}^{l}\rho^{(0)}(y)\cosh\sqrt{\alpha^2-1}\; k_0 y dy \tag{B.5}$$

$$J_0(\alpha)\left\{2\pi L_s + \int_{-l}^{l}\int_{-l}^{l}\Delta G_m(y-y')J_x^{(0)}(y')J_x^{(0)}(y)dy'dy\right\} = \alpha c_0\left[\int_{-l}^{l} J_x^{(0)}(y)\cosh\sqrt{\alpha^2-1}\; k_0 y dy + \rho_0(\alpha)\int_{-l}^{l}\int_{-l}^{l} M(y-y')\rho^{(0)}(y')J_x^{(0)}(y)dy'dy\right]. \tag{B.6}$$

These equations determine $\rho_0(\alpha)$ and $J_0(\alpha)$ in terms of the static strip parameters and some integral terms, of which the latter determine the frequency dependence. Since (7) and (15) give us

$$\alpha J_0(\alpha) = c_0\rho_0(\alpha) \tag{B.7}$$

we obtain an equation to solve for α.

The integrals in (B.5) and (B.6) can be simplified by introducing the Fourier transform pair (for functions which vanish for $|y|>l$)

$$\left.\begin{aligned} f(y) &= \int_{-\infty}^{\infty}\exp(-ik_0\lambda y)\tilde{f}(\lambda)d\lambda \\ \tilde{f}(\lambda) &= \frac{k_0}{2\pi}\int_{-l}^{l}\exp(ik_0\lambda y)f(y)dy \end{aligned}\right\}. \tag{B.8}$$

For even functions, the exponentials in (B.8) can be replaced by $\cos(k_0\lambda y)$. Since the static distributions are even, we get

$$\int_{-l}^{l}\rho^{(0)}(y)\cosh\sqrt{\alpha^2-1}\; k_0 y dy = \tilde{\rho}^{(0)}(-i\sqrt{\alpha^2-1}) \tag{B.9}$$

$$\int_{-l}^{l} J_x^{(0)}(y)\cosh\sqrt{\alpha^2-1}\; k_0 y dy = \tilde{J}_x^{(0)}(-i\sqrt{\alpha^2-1}) \tag{B.10}$$

while the double integrals in (B.5) and (B.6) can be reduced by using the Fourier integral representations for ΔG_e, ΔG_m, and M (cf. (2), (5), (6), (8) and (9)):

$$\frac{1}{2\pi}\int_{-l}^{l}\int_{-l}^{l}\Delta G_e(y-y')\rho^{(0)}(y')\rho^{(0)}(y)dy'dy \equiv G_e^{(2)}(\alpha) = \frac{4\pi}{k_0^2}\int_0^{\infty}\left[\frac{u_n\tanh u_n T}{u_0(\epsilon_r u_0 + u_n\tanh u_n T)} - \frac{\tanh\lambda T}{\lambda(\epsilon_r + \tanh\lambda T)}\right]\cdot\left[\tilde{\rho}^{(0)}(\lambda)\right]^2 d\lambda \tag{B.11}$$

$$\frac{1}{2\pi}\int_{-l}^{l}\int_{-l}^{l}\Delta G_m(y-y')J_x^{(0)}(y')J_x^{(0)}(y)dy'dy \equiv G_m^{(2)}(\alpha) = \frac{4\pi}{k_0^2}\cdot\int_0^{\infty}\left[\frac{1}{\mu_r u_0 + u_n\coth u_n T} - \frac{1}{\lambda(\mu_r + \coth\lambda T)}\right]\left[\tilde{J}_x^{(0)}(\lambda)\right]^2 d\lambda \tag{B.12}$$

$$\frac{1}{2\pi}\int_{-l}^{l}\int_{-l}^{l} M(y-y')\rho^{(0)}(y')J_x^{(0)}(y)dy'dy \equiv M^{(2)}(\alpha) = \frac{4\pi}{k_0^2}(\mu_r\epsilon_r - 1)\cdot\int_0^{\infty}\frac{\left[\tilde{\rho}^{(0)}(\lambda)\right]\left[\tilde{J}_x^{(0)}(\lambda)\right]d\lambda}{(\epsilon_r u_0 + u_n\tanh u_n T)(\mu_r u_0 + u_n\coth u_n T)u_0}. \tag{B.13}$$

Inserting (B.11)–(B.13) into (B.5)–(B.7), and taking (B.9) and (B.10) into account yields the eigenvalue equation for α:

$$\alpha^2\left[\frac{1}{C_s} + G_e^{(2)}(\alpha) + \frac{\tilde{\rho}^{(0)}(-i\sqrt{\alpha^2-1})}{\tilde{J}_x^{(0)}(-i\sqrt{\alpha^2-1})}M^{(2)}(\alpha)\right] = \left[L_s + G_m^{(2)}(\alpha)\right]\frac{\tilde{\rho}^{(0)}(-i\sqrt{\alpha^2-1})}{\tilde{J}_x^{(0)}(-i\sqrt{\alpha^2-1})}. \tag{B.14}$$

Although no exact, closed-form expressions for $\rho^{(0)}(y)$ and $J_x^{(0)}(y)$ exist, the following simple expressions for the case $\mu_r = 1$ have been obtained in [10]:

$$\rho^{(0)}(y) \simeq \frac{\text{const}}{\sqrt{\cosh^2(\pi l/2h) - \cosh^2(\pi y/2h)}}, \qquad |y| \leqslant l \tag{B.15}$$

$$J_x^{(0)}(y) \simeq \frac{\text{const}}{\sqrt{\cosh^2(\pi l/4t) - \cosh^2(\pi y/4t)}}, \qquad |y| \leqslant l. \tag{B.16}$$

Here $h = (\epsilon_r + 1)t/\epsilon_r$, which reduces approximately to t when $\epsilon_r \gg 1$. Expressions (B.15) and (B.16) have been found to be accurate to within a few percent for most parameter values, and have never been observed to deviate more than about 10 percent for any situation. These forms are particularly suited to our present purpose, because their Fourier transform can be expressed in terms of a particular form of the Legendre function known as a conical function [19, p. 14]:

$$P_{-1/2+i\tau}\left(\cosh\frac{\pi l}{h}\right)=\frac{1}{2h}\int_{-l}^{l}\frac{\cos k_0\lambda y\,dy}{\sqrt{\cosh^2(\pi l/2h)-\cosh^2(\pi y/2h)}} \tag{B.17}$$

where $\tau=\lambda k_0 h/\pi$. Efficient numerical procedures exist for computing this function: a uniform asymptotic expansion for large τ[19, p. 23], [21, p. 466], [22]; and for small τ either a power series whose coefficients are tabulated as functions of the argument $z=\cosh(\pi l/h)$ [19] or a method similar to the arithmetic–geometric mean algorithm for evaluating elliptic integrals [23].

By (12) and (13) (since $C_s V$ and I are both unity), we see that $\tilde{\rho}^{(0)}(0)=\tilde{J}_x^{(0)}(0)=k_0/2\pi$, which suffices to determine the constants in (B.15) and (B.16), so that by (B.17):

$$\tilde{\rho}^{(0)}(\lambda)=\frac{k_0}{2\pi}\frac{P_{-1/2+i\tau_e}(\cosh\pi l/h)}{P_{-1/2}(\cosh\pi l/h)} \tag{B.18}$$

$$\tilde{J}_x^{(0)}(\lambda)=\frac{k_0}{2\pi}\frac{P_{-1/2+i\tau_m}(\cosh\pi l/2t)}{P_{-1/2}(\cosh\pi l/2t)} \tag{B.19}$$

where $\tau_e=\lambda k_0 h/\pi=\lambda T(\epsilon_r+1)/\pi\epsilon_r$ and $\tau_m=2\lambda T/\pi$. By [19] it is possible to express $P_{-1/2}$ in terms of elliptic integrals [17]:

$$P_{-1/2}\left(\cosh\frac{\pi l}{h}\right)=\frac{2}{\pi}k_e'K(k_e)$$
$$P_{-1/2}\left(\cosh\frac{\pi l}{2t}\right)=\frac{2}{\pi}k_m'K(k_m) \tag{B.20}$$

where the moduli are given by

$$k_e=\tanh\frac{\pi l}{2h}\qquad k_e'=(1-k_e^2)^{1/2} \tag{B.21}$$
$$k_m=\tanh\frac{\pi l}{4t}\qquad k_m'=(1-k_m^2)^{1/2}.$$

Inserting (B.18) and (B.19) into (B.14), we arrive at dispersion relation (16)–(18), wherein

$$\frac{1}{C_d(\alpha)}=G_e^{(2)}(\alpha)+\frac{P_{-1/2}(\cosh\pi l/2t)}{P_{-1/2}(\cosh\pi l/h)}\frac{P_{\nu_e-1/2}(\cosh\pi l/2t)}{P_{\nu_m-1/2}(\cosh\pi l/2t)}M^{(2)}(\alpha) \tag{B.22}$$

$$L_d(\alpha)=L_s\left[\frac{P_{-1/2}(\cosh\pi l/2t)}{P_{-1/2}(\cosh\pi l/h)}\frac{P_{\nu_e-1/2}(\cosh\pi l/h)}{P_{\nu_m-1/2}(\cosh\pi l/2t)}-1\right]+\frac{P_{-1/2}(\cosh\pi l/2t)}{P_{-1/2}(\cosh\pi l/h)}\frac{P_{\nu_e-1/2}(\cosh\pi l/h)}{P_{\nu_m-1/2}(\cosh\pi l/2t)}G_m^{(2)}(\alpha) \tag{B.23}$$

where

$$\nu_e=k_0h\sqrt{\alpha^2-1}\,/\pi\text{ and }\nu_m=2k_0t\sqrt{\alpha^2-1}\,/\pi$$

and the Sommerfeld integrals are

$$G_e^{(2)}(\alpha)=\frac{1}{\pi}\int_0^\infty\left[\frac{u_n\tanh u_nT}{u_0(\epsilon_r u_0+u_n\tanh u_nT)}-\frac{\tanh\lambda T}{\lambda(\epsilon_r+\tanh\lambda T)}\right]\cdot\left[\frac{P_{-1/2+i\tau_e}(\cosh\pi l/h)}{P_{-1/2}(\cosh\pi l/h)}\right]^2 d\lambda \tag{B.24}$$

$$G_m^{(2)}(\alpha)=\frac{1}{\pi}\int_0^\infty\left[\frac{1}{\mu_r u_0+u_n\coth u_nT}-\frac{1}{\lambda(\mu+\coth\lambda T)}\right]\cdot\left[\frac{P_{-1/2+i\tau_m}(\cosh\pi l/2t)}{P_{-1/2}(\cosh\pi l/2t)}\right]^2 d\lambda \tag{B.25}$$

$$M^{(2)}(\alpha)=\frac{\mu_r\epsilon_r-1}{\pi}\cdot\int_0^\infty\left[\frac{1}{(\epsilon_r u_0+u_n\tanh u_nT)(\mu_r u_0+u_n\coth u_nT)u_0}\right]\cdot\left[\frac{P_{-1/2+i\tau_e}(\cosh\pi l/h)}{P_{-1/2}(\cosh\pi l/h)}\right]\cdot\left[\frac{P_{-1/2+i\tau_m}(\cosh\pi l/2t)}{P_{-1/2}(\cosh\pi l/2t)}\right]d\lambda. \tag{B.26}$$

Equations (B.24)–(B.26) differ from the corresponding functions in [1] only by an unimportant constant and the presence of $\tilde{\rho}^{(0)}(\lambda)$ and $\tilde{J}_x^{(0)}(\lambda)$ in the integrals.

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Short Papers

An Expansion for the Fringing Capacitance

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Abstract—**The first twelve terms in an expansion of the "approximate fringing capacitance" in powers of exp $(-\pi s/b)$ are given explicitly as functions of t/b. Comparison with exact values shows agreement within 0.06 percent for $s/b > 0.2$ and $t/b < 0.5$. In the extreme case considered, $s/b = 0.1$ and $t/b = 0.5$, the error is less than 2.3 percent.**

Introduction

The "approximate fringing capacitance" C_{f_0}', as defined by Cohn [1] and Getsinger [2] is useful in a number of ways in the approximation of the capacitance of certain rectangular coaxial structures. Explicit formulas for it have been given by Cockroft [3], Getsinger [2], and Riblet [4]. These formulas express C_{f_0}' in terms of two independent real parameters a and k. The normalized geometric parameters, t/b and s/b of Fig. 1 are also given in terms of these parameters, but, before C_{f_0}' can be found for a given geometry, these equations must be inverted in some way and a and k determined for the given values of t/b and s/b.

Heretofore this determination has required some form of graphical or numerical trial and error process. Recently, Riblet [5], however, has shown how for large values of s/b, these equations can be inverted. In this note these values for a and k are substituted directly in the formula for C_{f_0}' and an expansion obtained for C_{f_0}' in powers of $\exp(-\pi s/b)$, whose coefficients are given functions of t/b, which has useful accuracy for s/b as small as 0.1.

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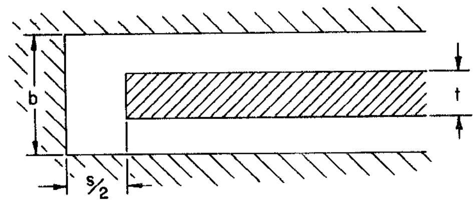

Fig. 1. Fringing capacitance cross section.

The Problem

It is not difficult, following Bowman [6] to express the quantities b, s, and t, of Fig. 1, except for a scale factor, in terms of two independent real parameters a and k, where k is the modulus of the Jacobi elliptic functions involved. It is no restriction to assume that $0 \leqslant k \leqslant 1$ and $0 \leqslant a \leqslant K$. Then

$$b = 2K\left\{\frac{\mathrm{sn}a\,\mathrm{dn}a}{\mathrm{cn}a} - Z(a)\right\} - \frac{\pi a}{K} + \pi \tag{1}$$

$$s = 2K\left\{\frac{\mathrm{sn}a\,\mathrm{dn}a}{\mathrm{cn}a} - Z(a)\right\} \tag{2}$$

$$t = 2K'\left\{\frac{\mathrm{sn}a\,\mathrm{dn}a}{\mathrm{cn}a} - Z(a)\right\} - \frac{\pi a}{K}. \tag{3}$$

The approximate odd-mode fringing capacitance, C_{f_0}' for this geometry is given in terms of the same parameters a and k by